Gravity and the Noncommutative Residue for Manifolds with Boundary *

Yong Wang

†*Center of Mathematical Sciences Zhejiang University Hangzhou Zhejiang 310027, China ;*

E-mail: wangy581@nenu.edu.cn

Abstract We prove a Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator for $3,4$-dimensional manifolds with boundary. As a corollary, we give two kinds of operator theoretic explanations of the gravitational action in the case of 4-dimensional manifolds with flat boundary.

Subj. Class.: Noncommutative global analysis; Noncommutative differential geometry.

MSC: 58G20; 53A30; 46L87

Keywords: Noncommutative residue for manifolds with boundary; gravitational action for manifolds with boundary

1 Introduction

The noncommutative residue found in [Gu] and [Wo] plays a prominent role in noncommutative geometry. In [C1],Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. In [C2], Connes proved that the noncommutative residue on a compact manifold M coincided with the Dixmier's trace on pseudodifferential operators of order $-\mathrm{dim}M$. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which we call the Kastler-Kalau-Walze theorem. In [K], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. In [A], Ackermann gave a note on a new proof of this theorem by means of the heat kernel expansion.

On the other hand, Fedosov et al defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [FGLS]. In [S], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. In [Wa1] and [Wa2], we generalized some results in [C1] and [U] to the case of manifolds with boundary . In [H], the gravitational action for

*Supported by Science Foundation for Young Teachers of Northeast Normal University (No. 20060102)

†also at School of Mathematics and Statistics, Northeast Normal University

manifolds with boundary was worked out (also see [B]). The motivation of this paper is to give an operator theoretic explanation of the gravitational action for manifolds with boundary and prove a Kastler-Kalau-Walze type theorem for manifolds with boundary.

Let us recall the Kastler-Kalau-Walze theorem in [K], [KW],[A]. Let M be a 4-dimensional oriented spin manifold (it holds for any even dimensional manifolds) and D be the associated Dirac operator on the spinor bundle $S(TM)$. Let s be the scalar curvature and Wres denote the noncommutative residue (see [Wo],[FGV]). Then the Kastler-Kalau-Walze theorem gives a spectral explanation of the gravitational action, it says that there exists a constant c_0, such that

$$\mathrm{Wres}(D^{-2}) = c_0 \int_M s\mathrm{dvol}_M. \tag{1.1}$$

For an oriented spin manifold M^4 with boundary ∂M, we use $\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2]$ instead of $\mathrm{Wres}(D^{-2})$ where $\widehat{D}$ is the Dirac operator on an open neighborhood $\widehat{M}$ of M and we still write D instead of $\widehat{D}$ in this paper (for definition of $\widehat{D}$ see Section 2) . Here $\widetilde{\mathrm{Wres}}$ denotes the noncommutative residue for manifolds with boundary of [FGLS] and π^+D^{-1} is an element in Boutet de Monvel's algebra (see [Wa1], Section 3). By definitions of Boutet de Monvel's algebra (see[S,p.11] or [Wa1,pp.5-6]), it is significant to consider

$$\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2] = \widetilde{\mathrm{Wres}}\left(\begin{bmatrix} \pi^+\widehat{D}^{-1} & 0 \\ 0 & 0 \end{bmatrix}^2\right), \tag{1.2}$$

which doesn't depend on the extension $\widehat{M}$. By the composition formula in Boutet de Monvel's algebra and the definition of $\widetilde{\mathrm{Wres}}$ (see (2.4) and (2.6) in [Wa1]), $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ is the sum of two terms one corresponding to interior and the other corresponding to boundary of M. It is well known that (see [H]) that the gravitational action for manifolds with boundary is also the sum of two terms from interior and boundary of M. So it is natural to hope to get the gravitational action for manifolds with boundary by computing $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$. For simplicity, we assume that the metric g^M on M has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \tag{1.3}$$

where $g^{\partial M}$ is the metric on ∂M. $h(x_n) \in C^\infty([0,1)) = \{\widetilde{h}|_{[0,1)} | \widetilde{h} \in C^\infty((-\varepsilon, 1))\}$ for some $\varepsilon > 0$ and satisfies $h(x_n) > 0,\ h(0) = 1$ where x_n denotes the normal directional coordinate. Through computations, we find that the term from boundary which we expect to get vanishes, so $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ is also proportional to $\int_M s\mathrm{dvol}_M$. Fortunately, if we assume that ∂M is flat, then we can define $\int_{\partial M} \mathrm{res}_{1,1}(D^{-1}, D^{-1})$ and $\int_{\partial M} \mathrm{res}_{2,1}(D^{-1}, D^{-1})$ (see Section 4) and get that the gravitational action for ∂M is proportional to $\int_{\partial M} \mathrm{res}_{1,1}(D^{-1}, D^{-1})$ and $\int_{\partial M} \mathrm{res}_{2,1}(D^{-1}, D^{-1})$, which gives two kinds of operator theoretic explanations of the gravitational action for boundary. For general even dimensional manifolds with boundary, we have no similar explanations

for the gravitational action for boundary, even though for the flat boundary (see Section 4).

For odd dimensional manifolds without boundary, $\mathrm{Wres}(D^{-2}) = 0$, so Kastler-Kalau-Walze Theorem isn't correct. But for odd dimensional manifolds with boundary, in general $\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2]$ doesn't vanish (similar to Section 5-7 in [Wa1]). In this paper we compute $\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2]$ explicitly for 3-dimensional spin manifolds with boundary.

This paper is organized as follows: In Section 2, for 4-dimensional spin manifolds with boundary and the associated Dirac operator D, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$. In Section 3, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ for 4-dimensional oriented Riemannian manifolds with boundary and the associated signature operator. Two kinds of operator theoretic explanations of the gravitational action for boundary in the case of 4-dimensional manifolds with boundary will be given in Section 4. In Section 5, We compute $\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2]$ for 3-dimensional spin manifolds with boundary. In Appendix, the proof of two facts in Section 2 will be given.

2 The Dirac operator case

In this section, we compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ by the brute force way in [K] and the normal coordinates way in [KW].

Let M be a n-dimensional compact oriented spin manifold with boundary ∂M and the metric g^M in (1.2). Let $n = 4$, but our some computations is correct for the general n. Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic to $\partial M \times [0, 1)$. By the definition of $C^\infty([0, 1))$ and $h > 0$, there exists $\widetilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\widetilde{h}|_{[0,1)} = h$ and $\widetilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\widehat{g}$ on $\widehat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\widehat{g} = \frac{1}{\widetilde{h}(x_n)} g^{\partial M} + dx_n^2, \tag{2.1}$$

such that $\widehat{g}|_M = g$. We fix a metric $\widehat{g}$ on the $\widehat{M}$ such that $\widehat{g}|_M = g$. We can get the spin structure on $\widehat{M}$ by extending the spin structure on M. Let D be the Dirac operator associated to $\widehat{g}$ on the spinors bundle $S(T\widehat{M})$. We want to compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ (for the related definitions, see [Wa1], Section 2, 3). Let $\mathbf{S}$ ($\mathbf{S}'$) be the unit sphere about ξ (ξ') and $\sigma(\xi)$ ($\sigma(\xi')$) be the corresponding canonical $n-1$ ($n-2$) volume form. Denote by $\sigma_l(A)$ the l- order symbol of an operator A. By (2.4) and (2.6) in [Wa1], we get

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2] = \widetilde{\mathrm{Wres}}[(\pi^+D^{-2}) + L(D^{-1}, D^{-1})]$$

$$= \int_M \int_{|\xi|=1} \mathrm{trace}_{S(TM)}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx + 2\pi \int_{\partial M} \int_{|\xi'|=1} \mathrm{tr}_{S(TM)}[\mathrm{tr}(b_{-4})(x', \xi')]\sigma(\xi')dx', \tag{2.2}$$

where b_{-4} is the (-4)-order symbol of $L(D^{-1}, D^{-1})$ which is called leftover term. By the formula (3.14) and (3.15) in [Wa1] and π'_{ξ_n} adding degree 1 of the symbol and

the ++ parts vanishing after integration with respect to ξ_n (see [FGLS] p. 23), then we have

$$\operatorname{tr}(b_{-4})=\sum_{j,k=0}^{\infty}\frac{(-i)^{j+k+1}}{(j+k+1)!}\pi'_{\xi_n}[\partial^j_{x_n}\partial^k_{\xi_n}a^+(x',0,\xi',\xi_n)\circ'\partial^{j+1}_{\xi_n}\partial^k_{x_n}a(x',0,\xi',\xi_n)]_{-4}, \tag{2.3}$$

where $a=\sigma(D^{-1})$ and π'_{ξ_n} and $a^+=\pi^+_{\xi_n}a$ defined by (2.1) and (2.2) in [Wa1]. By the formula of p.740 line 2 in [Wa1], we get

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]=\int_M\int_{|\xi|=1}\mathrm{trace}_{S(TM)}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx+\int_{\partial M}\Phi, \tag{2.4}$$

where

$$\Phi=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}$$

$$\times\mathrm{trace}_{S(TM)}[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r(D^{-1})(x',0,\xi',\xi_n)\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l(D^{-1})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{2.5}$$

where the sum is taken over $r-k-|\alpha|+l-j-1=-4,\ \ r,l\leq-1$. Since $[\sigma_{-4}(D^{-2})]|_M$ has the same expression as $\sigma_{-4}(D^{-2})$ in the case of manifolds without boundary, so locally we can use the computations in [K], [KW], [A], then we have

$$\int_M\int_{|\xi|=1}\operatorname{tr}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx=-\frac{\Omega_4}{3}\int_M s\mathrm{dvol}_M. \tag{2.6}$$

where $\Omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. So we only need to compute $\int_{\partial M}\Phi$.

Firstly, we compute the symbol $\sigma(D^{-1})$ of D^{-1}. Recall the definition of the Dirac operator D (see [BGV], [Y]). Let ∇^L denote the Levi-civita connection about g^M. In the local coordinates $\{x_i;1\leq i\leq n\}$ and the fixed orthonormal frame $\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1},\cdots,\widetilde{e_n})=(\widetilde{e_1},\cdots,\widetilde{e_n})(\omega_{s,t}). \tag{2.7}$$

$c(\widetilde{e_i})$ denotes the Clifford action. The Dirac operator

$$D=\sum_{i=1}^n c(\widetilde{e_i})[\widetilde{e_i}-\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t})]. \tag{2.8}$$

So we get,

$$\sigma_1(D)=\sqrt{-1}c(\xi);\sigma_0(D)=-\frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}), \tag{2.9}$$

where $\xi=\sum_{i=1}^n\xi_i dx_i$ denotes the cotangent vector. Write

$$D^\alpha_x=(-\sqrt{-1})^{|\alpha|}\partial^\alpha_x;\ \sigma(D)=p_1+p_0;\ \sigma(D^{-1})=\sum_{j=1}^{\infty}q_{-j}. \tag{2.10}$$

By the composition formula of psudodifferential operators, then we have

$$\begin{aligned}1=\sigma(D\circ D^{-1}) &= \sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}[\sigma(D)]D_{x}^{\alpha}[\sigma(D^{-1})]\\ &= (p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots)\\ &\quad +\sum_j(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots)\\ &= p_1q_{-1}+(p_1q_{-2}+p_0q_{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}q_{-1})+\cdots,\end{aligned}$$

Thus, we get:

$$q_{-1}=p_1^{-1};\ q_{-2}=-p_1^{-1}[p_0p_1^{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}(p_1^{-1})]. \tag{2.11}$$

By (2.9), (2.11) and direct computations, we have

Lemma 2.1

$$q_{-1}=\frac{\sqrt{-1}c(\xi)}{|\xi|^2};\quad q_{-2}=\frac{c(\xi)p_0c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)[\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2)] \tag{2.12}$$

Since Φ is a global form on ∂M, so for any fixed point $x_0\in\partial M$, we can choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of g^M on $\widetilde{U}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g_{ij}^M=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g^{ij}_M=g^M(dx_i,dx_j)$, then

$$[g^M_{i,j}]=\begin{bmatrix}\frac{1}{h(x_n)}[g^{\partial M}_{i,j}] & 0\\ 0 & 1\end{bmatrix};\quad [g_M^{i,j}]=\begin{bmatrix}h(x_n)[g^{i,j}_{\partial M}] & 0\\ 0 & 1\end{bmatrix}, \tag{2.13}$$

and

$$\partial_{x_s}g_{ij}^{\partial M}(x_0)=0,1\leq i,j\leq n-1;\quad g_{ij}^M(x_0)=\delta_{ij}. \tag{2.14}$$

Let $n=4$ and $\{e_1,\cdots,e_{n-1}\}$ be an orthonormal frame field in U about $g^{\partial M}$ which is parallel along geodesics and $e_i(x_0)=\frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1}=\sqrt{h(x_n)}e_1,\cdots,\widetilde{e_{n-1}}=\sqrt{h(x_n)}e_{n-1},\widetilde{e_n}=dx_n\}$ is the orthonormal frame field in $\widetilde{U}$ about g^M. Locally $S(TM)|_{\widetilde{U}}\cong\widetilde{U}\times\wedge^*_{\mathbf{C}}(\frac{n}{2})$. Let $\{f_1,\cdots,f_4\}$ be the orthonormal basis of $\wedge^*_{\mathbf{C}}(\frac{n}{2})$. Take a spin frame field $\sigma:\ \widetilde{U}\rightarrow\mathrm{Spin}(M)$ such that $\pi\sigma=\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$ where $\pi:\ \mathrm{Spin}(M)\rightarrow O(M)$ is a double covering, then $\{[(\sigma,f_i)],\ 1\leq i\leq 4\}$ is an orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form Φ is independent of the choice of the local frame, so we can compute $\mathrm{tr}_{S(TM)}$ in the frame $\{[(\sigma,f_i)],\ 1\leq i\leq 4\}$. Let $\{E_1,\cdots,E_n\}$ be the canonical basis of $\mathbf{R}^n$ and $c(E_i)\in\mathrm{cl}_{\mathbf{C}}(n)\cong\mathrm{Hom}(\wedge^*_{\mathbf{C}}(\frac{n}{2}),\wedge^*_{\mathbf{C}}(\frac{n}{2}))$ be the Clifford action. By [Y], then

$$c(\widetilde{e_i})=[(\sigma,c(E_i))];\ c(\widetilde{e_i})[(\sigma,f_i)]=[(\sigma,c(E_i)f_i)];\ \frac{\partial}{\partial x_i}=[(\sigma,\frac{\partial}{\partial x_i})], \tag{2.15}$$

then we have $\frac{\partial}{\partial x_i}c(\widetilde{e_i})=0$ in the above frame.

Lemma 2.2 $\partial_{x_j}(|\xi|^2_{g^M})(x_0)=0,$ if $j<n;\ =h'(0)|\xi'|^2_{g^{\partial M}},$ if $j=n.$ (2.16)
$\partial_{x_j}[c(\xi)](x_0)=0,$ if $j<n;\ =\partial_{x_n}[c(\xi')](x_0),$ if $j=n,$ (2.17)
where $\xi=\xi'+\xi_n dx_n$.
Proof. By the equality $\partial_{x_j}(|\xi|^2_{g^M})(x_0)=\partial_{x_j}(h(x_n)g^{l,m}_{\partial M}(x')\xi_l\xi_m+\xi_n^2)(x_0)$ and (2.14), then (2.16) is correct. By Lemma A.1 in Appendix, (2.17) is correct. □

In order to compute $p_0(x_0)$, we need to compute $\omega_{s,t}(\widetilde{e_i})(x_0)$.

Lemma 2.3 *When* $i<n,\ \omega_{n,i}(\widetilde{e_i})(x_0)=\frac{1}{2}h'(0);$ *and* $\omega_{i,n}(\widetilde{e_i})(x_0)=-\frac{1}{2}h'(0),$ *In other cases,* $\omega_{s,t}(\widetilde{e_i})(x_0)=0$
Proof. See Appendix. □

Lemma 2.4 $p_0(x_0)=c_0c(dx_n)$, where $c_0=-\frac{3}{4}h'(0)$.
Proof. This comes from (2.9), Lemma 2.3 and the relation $c(\widetilde{e_i})c(\widetilde{e_j})+c(\widetilde{e_j})c(\widetilde{e_i})=-2\delta_{i.j}$. □

Now we can compute Φ, since the sum is taken over $-r-l+k+j+|\alpha|=-3,\ r,l\leq -1$, then we have the following five cases:

case a) I) $r=-1,\ l=-1\ k=j=0,\ |\alpha|=1$

By (2.5), we get

$$\text{case a) I)}=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}q_{-1}\times\partial^\alpha_{x'}\partial_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \quad (2.17)$$

By Lemma 2.2, for $i<n$, then

$$\partial_{x_i}q_{-1}(x_0)=\partial_{x_i}\left(\frac{\sqrt{-1}c(\xi)}{|\xi|^2}\right)(x_0)=\frac{\sqrt{-1}\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{\sqrt{-1}c(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}=0,$$

so case a) I) vanishes.

case a) II) $r=-1,\ l=-1\ k=|\alpha|=0,\ j=1$

By (2.5), we get

$$\text{casea) II)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}q_{-1}\times\partial^2_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \quad (2.18)$$

By Lemma 2.1 and Lemma 2.2, we have

$$\partial^2_{\xi_n}q_{-1}=\sqrt{-1}\left(-\frac{6\xi_nc(dx_n)+2c(\xi')}{|\xi|^4}+\frac{8\xi_n^2c(\xi)}{|\xi|^6}\right); \quad (2.19)$$

$$\partial_{x_n}q_{-1}(x_0)=\frac{\sqrt{-1}\partial_{x_n}c(\xi')(x_0)}{|\xi|^2}-\frac{\sqrt{-1}c(\xi)|\xi'|^2h'(0)}{|\xi|^4}. \quad (2.20)$$

By (2.1) in [Wa1] and the Cauchy integral formula, then

$$\begin{aligned}
\pi^+_{\xi_n}\left[\frac{c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1} &= \pi^+_{\xi_n}\left[\frac{c(\xi')+\xi_nc(dx_n)}{(1+\xi_n^2)^2}\right] \\
&= \frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{\frac{c(\xi')+\eta_nc(dx_n)}{(\eta_n+i)^2(\xi_n+iu-\eta_n)}}{(\eta_n-i)^2}d\eta_n \\
&= \left[\frac{c(\xi')+\eta_nc(dx_n)}{(\eta_n+i)^2(\xi_n-\eta_n)}\right]^{(1)}|_{\eta_n=i} \\
&= -\frac{ic(\xi')}{4(\xi_n-i)}-\frac{c(\xi')+ic(dx_n)}{4(\xi_n-i)^2} \qquad (2.21)
\end{aligned}$$

Similarly,

$$\pi^+_{\xi_n}\left[\frac{\sqrt{-1}\partial_{x_n}c(\xi')}{|\xi|^2}\right](x_0)|_{|\xi'|=1}=\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}. \qquad (2.22)$$

By (2.20), (2.21), (2.22), then

$$\pi^+_{\xi_n}\partial_{x_n}q_{-1}(x_0)|_{|\xi'|=1}=\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}+\sqrt{-1}h'(0)\left[\frac{ic(\xi')}{4(\xi_n-i)}+\frac{c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right]. \qquad (2.23)$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then we have the equalities:

$$\begin{aligned}
&\mathrm{tr}[c(\xi')c(dx_n)]=0;\quad \mathrm{tr}[c(dx_n)^2]=-4;\quad \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1}=-4;\\
&\mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)]=0;\quad \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-2h'(0). \qquad (2.24)
\end{aligned}$$

By (2.24) and direct computations ,we have

$$h'(0)\mathrm{tr}\left\{\left[\frac{ic(\xi')}{4(\xi_n-i)}+\frac{c(\xi')+ic(dx_n)}{4(\xi_n-i)^2}\right]\times\left[\frac{6\xi_nc(dx_n)+2c(\xi')}{(1+\xi_n^2)^2}-\frac{8\xi_n^2[c(\xi')+\xi_nc(dx_n)]}{(1+\xi_n^2)^3}\right]\right\}(x_0)|_{|\xi'|=1}$$

$$=-4h'(0)\frac{-2i\xi_n^2-\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}. \qquad (2.25)$$

Similarly, we have

$$-\sqrt{-1}\mathrm{tr}\left\{\left[\frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n-i)}\right]\times\left[\frac{6\xi_nc(dx_n)+2c(\xi')}{(1+\xi_n^2)^2}-\frac{8\xi_n^2[c(\xi')+\xi_nc(dx_n)]}{(1+\xi_n^2)^3}\right]\right\}(x_0)|_{|\xi'|=1}$$

$$=-2\sqrt{-1}h'(0)\frac{3\xi_n^2-1}{(\xi_n-i)^4(\xi_n+i)^3}. \qquad (2.26)$$

By (2.19), (2.23), (2.25), (2.26), then

$$\text{case a) II)} \;=\; -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{ih'(0)(\xi_n-i)^2}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma(\xi')dx'$$

$$
\begin{aligned}
&= -ih'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^2(\xi_n+i)^3}d\xi_n dx' \\
&= -ih'(0)\Omega_3 2\pi i[\frac{1}{(\xi_n+i)^3}]^{(1)}|_{\xi_n=i}dx' \\
&= -\frac{3}{8}\pi h'(0)\Omega_3 dx'.
\end{aligned}
$$

case a) III) $r=-1,\ l=-1\ j=|\alpha|=0,\ k=1$

By (2.5), we get

$$
\text{case a) III)} = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}q_{-1}\times\partial_{\xi_n}\partial_{x_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \quad (2.27)
$$

By Lemma 2.2, we have

$$
\partial_{\xi_n}\partial_{x_n}q_{-1}(x_0)|_{|\xi'|=1} = -\sqrt{-1}h'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]-\frac{2\xi_n\sqrt{-1}\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}. \quad (2.28)
$$

$$
\partial_{\xi_n}\pi^+_{\xi_n}q_{-1}(x_0)|_{|\xi'|=1} = -\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}. \quad (2.29)
$$

Similarly to (2.25), (2.26), we have

$$
\begin{aligned}
\mathrm{tr}\left\{\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\sqrt{-1}h'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c(\xi')+\xi_n c(dx_n)}{|\xi|^6}\right]\right\} \\
= 2h'(0)\frac{i-3\xi_n}{(\xi_n-i)^4(\xi_n+i)^3}; \quad (2.30)
\end{aligned}
$$

and

$$
\mathrm{tr}\left[\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}\times\frac{2\xi_n\sqrt{-1}\partial_{x_n}c(\xi')(x_0)}{|\xi|^4}\right] = -2h'(0)\sqrt{-1}\frac{\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}. \quad (2.31)
$$

So we get case a) III)$=\frac{3}{8}\pi h'(0)\Omega_3 dx'$.

case b) $r=-2,\ l=-1,\ k=j=|\alpha|=0$

By (2.5), we get

$$
\text{case b)} = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}q_{-2}\times\partial_{\xi_n}q_{-1}](x_0)d\xi_n\sigma(\xi')dx', \quad (2.32)
$$

By Lemma 2.1 and Lemma 2.2, we have

$$
q_{-2}(x_0) = \frac{c(\xi)p_0(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)[\partial_{x_n}[c(\xi')](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial M}]. \quad (2.33)
$$

Then

$$\pi^+_{\xi_n} q_{-2}(x_0)|_{|\xi'|=1} = \pi^+_{\xi_n}\left[\frac{c(\xi)p_0(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0)}{(1+\xi_n^2)^2}\right]$$

$$-h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\right] := B_1 - B_2. \tag{2.34}$$

Similarly to (2.21), we have

$$B_1 = -\frac{A_1}{4(\xi_n - i)} - \frac{A_2}{4(\xi_n - i)^2}, \tag{2.35}$$

where

$$A_1 = ic(\xi')p_0c(\xi') + ic(dx_n)p_0c(dx_n) + ic(\xi')c(dx_n)\partial_{x_n}[c(\xi')];$$

$$A_2 = [c(\xi') + ic(dx_n)]p_0[c(\xi') + ic(dx_n)] + c(\xi')c(dx_n)\partial_{x_n}c(\xi') - i\partial_{x_n}[c(\xi')]. \tag{2.36}$$

$$\begin{aligned} B_2 &= h'(0)\pi^+_{\xi_n}\left[\frac{-\xi_n^2 c(dx_n)^2 - 2\xi_n c(\xi') + c(dx_n)}{(1+\xi_n^2)^3}\right] \\ &= \frac{h'(0)}{2}\left[\frac{-\eta_n^2 c(dx_n) - 2\eta_n c(\xi') + c(dx_n)}{(\eta_n + i)^3(\xi_n - \eta_n)}\right]^{(2)}|_{\eta_n = i} \\ &= \frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}[ic(\xi') - c(dx_n)]\right]. \end{aligned} \tag{2.37}$$

$$\partial_{\xi_n} q_{-1}(x_0)|_{|\xi'|=1} = \sqrt{-1}\left[\frac{c(dx_n)}{1+\xi_n^2} - \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1+\xi_n^2)^2}\right]. \tag{2.38}$$

By (2.37), (2.38), we have

$$\mathrm{tr}[B_2 \times \partial_{\xi_n} q_{-1}(x_0)]|_{|\xi'|=1} = \frac{\sqrt{-1}}{2}h'(0)\mathrm{trace}$$

$$\Big\{\{\left[\frac{1}{4i(\xi_n - i)} + \frac{1}{8(\xi_n - i)^2} - \frac{3\xi_n - 7i}{8(\xi_n - i)^3}\right]c(dx_n) + \left[\frac{-1}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}\right]ic(\xi')\}$$

$$\times\{\left[\frac{1}{1+\xi_n^2} - \frac{2\xi_n^2}{(1+\xi_n^2)^2}\right]c(dx_n) - \frac{2\xi_n}{(1+\xi_n^2)^2}c(\xi')\}\Big\}$$

$$= \frac{\sqrt{-1}}{2}h'(0)\frac{-i\xi_n^2 - \xi_n + 4i}{4(\xi_n - i)^3(\xi_n + i)^2}\mathrm{tr}[\mathrm{id}]. \tag{2.39}$$

Note that

$$B_1 = \frac{-1}{4(\xi_n - i)^2}[(2 + i\xi_n)c(\xi')p_0c(\xi') + i\xi_n c(dx_n)p_0c(dx_n)$$

$$+(2 + i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)p_0c(\xi') + ic(\xi')p_0c(dx_n) - i\partial_{x_n}c(\xi')]. \tag{2.40}$$

By (2.24), (2.38), (2.40), Lemma 2.4 and $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, considering for $i<n$ $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0$, then

$$\mathrm{tr}[B_1\times\partial_{\xi_n}q_{-1}(x_0)]|_{|\xi'|=1}=\frac{-2ic_0}{(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2}. \tag{2.41}$$

By (2.34), (2.39) and (2.41), we have

$$\text{case b)}=-\Omega_3\int_{\Gamma_+}\frac{2c_0(\xi_n-i)+ih'(0)}{(\xi_n-i)^3(\xi_n+i)^2}d\xi_n dx'=\frac{9}{8}\pi h'(0)\Omega_3 dx'. \tag{2.42}$$

case c) $r=-1,\ l=-2,\ k=j=|\alpha|=0$

By (2.5), we get

$$\text{case c)}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}q_{-1}\times\partial_{\xi_n}q_{-2}](x_0)d\xi_n\sigma(\xi')dx'. \tag{2.43}$$

By

$$\pi^+_{\xi_n}q_{-1}(x_0)|_{|\xi'|=1}=\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)}; \tag{2.44}$$

$$\partial_{\xi_n}q_{-2}(x_0)|_{|\xi'|=1}=\frac{1}{(1+\xi_n^2)^3}[(2\xi_n-2\xi_n^3)c(dx_n)p_0c(dx_n)+(1-3\xi_n^2)c(dx_n)p_0c(\xi')$$
$$+(1-3\xi_n^2)c(\xi')p_0c(dx_n)-4\xi_nc(\xi')p_0c(\xi')+(3\xi_n^2-1)\partial_{x_n}c(\xi')-4\xi_nc(\xi')c(dx_n)\partial_{x_n}c(\xi')$$
$$+2h'(0)c(\xi')+2h'(0)\xi_nc(dx_n)]+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n^2)^4}, \tag{2.45}$$

then similarly to computations of the case b), we have

$$\mathrm{trace}[\pi^+_{\xi_n}q_{-1}\times\partial_{\xi_n}q_{-2}](x_0)|_{|\xi'|=1}=\frac{3h'(0)(i\xi_n^2+\xi_n-2i)}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{12h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}. \tag{2.46}$$

So case c) $=-\frac{9}{8}\pi h'(0)\Omega_3dx'$. Now Φ is the sum of the cases a), b) and c), so is zero. Then we get

Theorem 2.5 *Let M be a 4-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above and D be the Dirac operator on $\widehat{M}$, then*

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]=-\frac{\Omega_4}{3}\int_M s\mathrm{dvol}_M. \tag{2.47}$$

Remark 2.6 Since (2.4) is correct for any dimensional manifolds with boundary, we conjecture that Theorem 2.5 is also correct for any even dimensional manifolds with boundary. But our computations way maybe isn't valid for general even dimensional manifolds with boundary. When the dimension becomes larger and larger, the terms which we need to compute becomes more and more. Maybe the way in [GS] is valid for any even dimensional manifolds with boundary.

3 The signature operator case

Let M be a 4-dimensional compact oriented Riemannian manifold with boundary ∂M and the metric in Section 2. $D = d+\delta: \ \wedge^*(T^*M) \to \wedge^*(T^*M)$ is the signature operator. Take the coordinates and the orthonormal frame as in Section 2. Let $\epsilon(\widetilde{e_j*}),\ \iota(\widetilde{e_j*})$ be the exterior and interior multiplications respectively. Write

$$c(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) - \iota(\widetilde{e_j*});\quad \bar{c}(\widetilde{e_j}) = \epsilon(\widetilde{e_j*}) + \iota(\widetilde{e_j*}). \tag{3.1}$$

We'll compute $\mathrm{tr}_{\wedge^*(T^*M)}$ in the frame $\{dx_{i_1}\wedge\cdots\wedge dx_{i_k} | \ 1\leq i_1 < \cdots < i_k \leq 4\}$. By [Y], we have

$$D = d+\delta = \sum_{i=1}^{n} c(\widetilde{e_i})[\widetilde{e_i} + \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]]. \tag{3.2}$$

So

$$p_1 = \sigma_1(d+\delta) = \sqrt{-1}c(\xi);\ p_0 = \sigma_0(d+\delta) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]. \tag{3.3}$$

Lemmas 2.1-2.3 are also correct, by Lemma 2.3, then

$$p_0(x_0) = \widetilde{p_0}(x_0) - \frac{3}{4}h'(0)c(dx_n);\ \widetilde{p_0}(x_0) = \frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(\widetilde{e_i})\bar{c}(\widetilde{e_n})\bar{c}(\widetilde{e_i})(x_0). \tag{3.4}$$

For the signature operator case,

$$\mathrm{tr}[\mathrm{id}] = 16;\quad \mathrm{tr}[c(\xi')\partial_{x_n}c(\xi')](x_0)|_{|\xi'|=1} = -8h'(0); \tag{3.5}$$

$$\mathrm{tr}[c(\xi')p_0c(\xi')c(dx_n)](x_0) = \mathrm{tr}[p_0c(\xi')c(dx_n)c(\xi')](x_0) = |\xi'|^2\mathrm{tr}[p_0c(dx_n)]. \tag{3.6}$$

$$\begin{aligned} c(dx_n)\widetilde{p_0}(x_0) &= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(e_i)\bar{c}(e_i)c(e_n)\bar{c}(e_n) \\ &= -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)] \end{aligned}$$

By Theorem 4.3 in [U], then

$$\begin{aligned} &\mathrm{tr}_{\wedge^m(T^*M)}\{[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)]\} \\ &= a_{n,m} < e_i*, e_n* >^2 + b_{n,m}|e_i*|^2|e_n*|^2 = b_{n,m}, \end{aligned} \tag{3.7}$$

where $b_{4,m} = \begin{pmatrix} 2 \\ m-2 \end{pmatrix} + \begin{pmatrix} 2 \\ m \end{pmatrix} - 2\begin{pmatrix} 2 \\ m-1 \end{pmatrix}$. By (3.7), then

$$\mathrm{tr}_{\wedge^*(T^*M)}\{[\epsilon(e_i*)\iota(e_i*) - \iota(e_i*)\epsilon(e_i*)][\epsilon(e_n*)\iota(e_n*) - \iota(e_n*)\epsilon(e_n*)]\} = \sum_{m=0}^{4} b_{4,m} = 0.$$

Then

$$\mathrm{tr}_{\wedge^*(T^*M)}[c(dx_n)\widetilde{p_0}(x_0)] = 0. \tag{3.8}$$

By (3.4), (3.5), (3.6) and (3.8), then $\Phi_{\rm sig} = 4\Phi_{\rm Dirac} = 0$. So we get

Theorem 3.1 *Let M be a 4-dimensional compact oriented Riemaniann manifold with the boundary ∂M and the metric g^M as above and D be the signature operator on $\widehat{M}$, then*

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2] = \frac{8\Omega_4}{3}\int_M s\mathrm{dvol}_M. \tag{3.9}$$

4 The gravitational action for 4-dimensional manifolds with boundary

Firstly, we recall the Einstein-Hilbert action for manifolds with boundary (see [H] or [B]),

$$I_{\rm Gr} = \frac{1}{16\pi}\int_M s\mathrm{dvol}_M + 2\int_{\partial M} K\mathrm{dvol}_{\partial_M} := I_{\rm Gr,i} + I_{\rm Gr,b}, \tag{4.1}$$

where

$$K = \sum_{1\leq i,j\leq n-1} K_{i,j}g^{i,j}_{\partial M};\quad K_{i,j} = -\Gamma^n_{i,j}, \tag{4.2}$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2, $K_{i,j}(x_0) = -\Gamma^n_{i,j}(x_0) = -\frac{1}{2}h'(0)$, when $i = j < n$, otherwise is zero. For $n = 4$, then

$$K(x_0) = \sum_{i,j} K_{i.j}(x_0)g^{i,j}_{\partial M}(x_0) = \sum_{i=1}^{3} K_{i,i}(x_0) = -\frac{3}{2}h'(0),$$

So

$$I_{\rm Gr,b} = -3h'(0)\mathrm{Vol}_{\partial M}. \tag{4.3}$$

Let M be 4-dimensional manifolds with boundary and P, P' be two pseudodifferential operators with transmission property (see [Wa1] or [RS]) on $\widehat{M}$. By (2.4) in [Wa1], we have

$$\pi^+P\circ\pi^+P' = \pi^+(PP') + L(P, P') \tag{4.4}$$

and $L(P, P')$ is leftover term which represents the difference between the composition $\pi^+P\circ\pi^+P'$ in Boutet de Monvel algebra and the composition PP' in the classical pseudodifferential operators algebra. By (2.5), we define locally

$$\mathrm{res}_{1,1}(P, P') := -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(P)\times\partial^2_{\xi_n}\sigma_{-1}(P')]d\xi_n\sigma(\xi')dx'; \tag{4.5}$$

$$\mathrm{res}_{2,1}(P, P') := -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(P)\times\partial_{\xi_n}\sigma_{-1}(P')]d\xi_n\sigma(\xi')dx'. \tag{4.6}$$

Thus they represent the difference between the composition $\pi^+P\circ\pi^+P'$ in Boutet de Monvel algebra and the composition PP' in the classical pseudodifferential operators algebra partially and

$$\text{case a) II)} = \mathrm{res}_{1,1}(D^{-1},D^{-1});\ \text{case b)} = \mathrm{res}_{2,1}(D^{-1},D^{-1}). \tag{4.7}$$

Now, we assume ∂M is flat , then $\{dx_i = e_i\}$, $g^{\partial M}_{i,j} = \delta_{i,j}$, $\partial_{x_s} g^{\partial M}_{i,j} = 0$. So $\mathrm{res}_{1,1}(D^{-1},D^{-1})$ and $\mathrm{res}_{2,1}(D^{-1},D^{-1})$ are two global forms locally defined by the aboved oriented orthonormal basis $\{dx_i\}$. By case a) II) and case b), then we have:

Theorem 4.1 *Let M be a 4-dimensional compact spin manifold with the boundary ∂M and the metric g^M as above and D be the Dirac operator on $\widehat{M}$. Assume ∂M is flat, then*

$$\int_{\partial M} \mathrm{res}_{1,1}(D^{-1},D^{-1}) = \frac{\pi}{8}\Omega_3 I_{\mathrm{Gr,b}}; \tag{4.8}$$

$$\int_{\partial M} \mathrm{res}_{2,1}(D^{-1},D^{-1}) = -\frac{3\pi}{8}\Omega_3 I_{\mathrm{Gr,b}}. \tag{4.9}$$

Theorem 4.2 *Let M be a 4-dimensional compact oriented Riemaniann manifold with the boundary ∂M and the metric g^M as above and D be the signature operator on $\widehat{M}$. Assume ∂M is flat, then*

$$\int_{\partial M} \mathrm{res}_{1,1}(D^{-1},D^{-1}) = \frac{\pi}{2}\Omega_3 I_{\mathrm{Gr,b}}; \tag{4.10}$$

$$\int_{\partial M} \mathrm{res}_{2,1}(D^{-1},D^{-1}) = -\frac{3\pi}{2}\Omega_3 I_{\mathrm{Gr,b}}. \tag{4.11}$$

Remark 4.3 We take N is a flat 3-dimensional oriented Riemannian manifold and $M = N\times[0,1]$, then $\partial M = N\oplus N$. Let $g^M = \frac{1}{h(x_n)}g^N + dx_n^2$, where $h(x_n) = 1 - x_n(x_n-1) > 0$ for $x_n\in[0,1]$ and $h(0)=h(1)=1$. The (M,g^M) satisfies conditions in Theorem 4.2. Similar construction is correct for Theorem 4.1. When ∂M is not connected, we still define the noncommutative residue with the loss of the unique property.

Remark 4.4 Considering (2.5). when the dimension increases, the degree of the derivative of $h(x_n)$ in Φ will increase. So the theorems 4.1 and 4.2 aren't correct for any even dimensional manifolds.

Remark 4.5 The reason that the term from boundary does not appear is perhaps that we ignore boundary conditions. We hope to compute the noncommutative residue $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ under certain boundary conditions to get the term from boundary in the future. Grubb and Schrohe got the noncommutative residue for manifolds with boundary through asymptotic expansions in [GS]. Another problem is to compute $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ by asymptotic expansions.

5 Computations of $\widetilde{\mathrm{Wres}}[(\pi^+\widehat{D}^{-1})^2]$ for 3-dimensional spin manifolds with boundary

For an odd dimensional manifolds with boundary, as in Section 5-7 in [Wa1], we have the formula

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2] = \int_{\partial M}\Phi. \tag{5.1}$$

When $n=3$, then in (2.5), $r-k-|\alpha|+l-j-1=-3,\quad r,l\leq -1$, so we get $r=l=-1,\ k=|\alpha|=j=0$, then

$$\Phi=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}_{S(TM)}[\sigma^+_{-1}(D^{-1})(x',0,\xi',\xi_n)\times\partial_{\xi_n}\sigma_{-1}(D^{-1})(x',0,\xi',\xi_n)]d\xi_3\sigma(\xi')dx'. \tag{5.2}$$

By Lemma 2.1, then similar to (2.21), we have

$$\sigma^+_{-1}(D^{-1})|_{|\xi'|=1}=\frac{\sqrt{-1}[c(\xi')+ic(dx_n)]}{2i(\xi_n-i)}; \tag{5.3}$$

$$\partial_{\xi_n}\sigma_{-1}(D^{-1})|_{|\xi'|=1}=\frac{\sqrt{-1}c(dx_n)}{1+\xi_n^2}-\frac{2\sqrt{-1}\xi_nc(\xi)}{(1+\xi_n^2)^2}. \tag{5.4}$$

For $n=3$, we take the coordinates as in Section 2. Locally $S(TM)|_{\widetilde{U}}\cong\widetilde{U}\times\wedge^{\mathrm{even}}_{\mathbf{C}}(2)$. Let $\{\widetilde{f_1},\widetilde{f_2}\}$ be an orthonormal basis of $\wedge^{\mathrm{even}}_{\mathbf{C}}(2)$ and we will compute the trace under this basis. Similarly to (2.24), we have

$$\mathrm{tr}[c(\xi')c(dx_3)]=0;\quad \mathrm{tr}[c(dx_3)^2]=-2;\quad \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1}=-2 \tag{5.5}$$

Then by (5.3) (5.4) and (5.5), we get

$$\mathrm{trace}[\sigma^+_{-1}(D^{-1})\times\partial_{\xi_n}\sigma_{-1}(D^{-1})](x_0)|_{|\xi'|=1}=-\frac{1}{(\xi_n+i)^2(\xi_n-i)}. \tag{5.6}$$

By (5.2) and (5.6) and the Cauchy integral formula, we get

$$\Phi=\frac{i\pi}{2}\Omega_2\mathrm{vol}_{\partial M}=i\pi^2\mathrm{vol}_{\partial M}. \tag{5.7}$$

Here $\mathrm{vol}_{\partial M}$ denotes the canonical volume form of ∂M.

Theorem 5.1 *Let M be a 3-dimensional compact spin manifold with the boundary ∂M and the metric g^M as in Section 2 and D be the Dirac operator on $\widehat{M}$, then*

$$\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]=i\pi^2\mathrm{Vol}_{\partial M}, \tag{5.8}$$

where $\mathrm{Vol}_{\partial M}$ denotes the canonical volume of ∂M.

Remark 5.2 By Theorem 5.1, we know that $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ isn't proportional to the gravitational action for boundary for 3-dimensional manifolds with boundary. By

the same reason as in Remark 4.4, we know that $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ isn't proportional to the gravitational action for boundary for any odd dimensional manifolds with boundary.

Appendix

In this appendix, we will prove some facts used in Lemma 2.2 and Lemma 2.3.

Lemma A.1

$$\partial_{x_l}c(dx_j)(x_0)=0, when\ l<n;\ \partial_{x_l}c(dx_n)=0$$

Proof. The fundamental setup is as in Section 2. Write $<\partial_{x_s},e_i>_{g^{\partial M}}=H_{i,s}$, then by [Y] or [BGV], $\partial_{x_j}H_{i,s}(x_0)=0$. Define $dx_j^*\in TM|_{\widetilde{U}}$ by $<dx_j^*,v>=(dx_j,v)$ for $v\in TM$. For $j<n$,

$$\begin{aligned} c(dx_j) &= c(dx_j^*)=c(\sum_i<dx_j^*,\widetilde{e_i}>\widetilde{e_i}) \\ &= \sum_{i,s}g^{s,j}<\partial_{x_s},\widetilde{e_i}>_{g^M}c(\widetilde{e_i})=\sum_{1\leq i,s<n}\frac{1}{\sqrt{h(x_n)}}g^{s,j}H_{s,i}c(\widetilde{e_i})+\sum_{i=s=n}g^{n,j}c(\widetilde{e_n}). \end{aligned}$$

So for $l<n$, $\partial_{x_l}c(dx_j)(x_0)=0$. □

The proof of Lemma 2.3:

Recall, let ∇^L be the Levi-Civita connection about g^M and

$$\nabla^L_{\partial_{x_i}}\partial_{x_j}=\sum_{k=1}^n\Gamma^k_{i,j}\partial_{x_k}, \tag{A.1}$$

then

$$\Gamma^k_{i,j}=\frac{1}{2}g^{kl}(\partial_{x_j}g_{li}+\partial_{x_i}g_{lj}-\partial_{x_l}g_{ij}). \tag{A.2}$$

Let

$$\partial_{x_i}=\sum_k h_{ik}\widetilde{e_k};\quad \widetilde{e_i}=\sum_k\widetilde{h_{ik}}\partial_{x_k}, \tag{A.3}$$

then the matrix $[h_{ik}]$ and $[\widetilde{h_{ik}}]$ are invertible, and $\widetilde{h_{ik}}(x_0)=\delta_{ik}$. By (A.1) and (A.3), then

$$\begin{aligned} \nabla^L_{\widetilde{e_i}}\widetilde{e_t}(x_0) &= \nabla^L_{\partial_{x_i}}(\sum_k\widetilde{h_{tk}}\partial_{x_k}) \\ &= \sum_k\partial_{x_i}(\widetilde{h_{tk}})\partial_{x_k}+\sum_{k,l}\widetilde{h_{t,k}}\Gamma^l_{ik}\partial_{x_l} \\ &= \sum_s[\sum_k\partial_{x_i}(\widetilde{h_{tk}})h_{ks}+\sum_{k,l}\widetilde{h_{t,k}}\Gamma^l_{ik}h_{ls}]\widetilde{e_s}. \end{aligned}$$

By (2.7), then

$$\omega_{st}(\widetilde{e_i})(x_0)=\partial_{x_i}(\widetilde{h_{ts}})(x_0)+\Gamma^s_{it}(x_0)=-\partial_{x_i}h_{ts}(x_0)+\Gamma^s_{it}(x_0). \tag{A.4}$$

By (A.2) and the choices of g^M and the normal coordinates of x_0 in ∂M, then we have

Lemma A.2 *When $i < n$, then*

$$\Gamma^n_{ii}(x_0) = \frac{1}{2}h'(0);\ \Gamma^i_{ni}(x_0) = -\frac{1}{2}h'(0);\ \Gamma^i_{in}(x_0) = -\frac{1}{2}h'(0),$$

in other cases, $\Gamma^i_{st}(x_0) = 0$.

By $h_{ts} = g^M(\partial_{x_t}, \widetilde{e_s}) = \frac{1}{\sqrt{h(x_n)}}H_{ts},\ (1 \leq t, s < n)$, then we have

Lemma A.3 $-\partial_{x_i}h_{ts}(x_0) = \frac{1}{2}h'(0)$ *if* $i = n,\ t = s < n$. *In other cases,* $-\partial_{x_i}h_{ts}(x_0) = 0$.

By Lemma A.2 and A.3 , (A.4), then we prove Lemma 2.3. □

Acknowledgement: The author is indebted to Professor Weiping Zhang for his encouragement and support. He thanks Professors Xianzhe Dai and Siye Wu for their helpful discussions on the gravitational action for manifolds with boundary.

References

[A] T. Ackermann, *A note on the Wodzicki residue,* J. Geom. Phys., 20, 404-406, 1996.

[B] N. H. Barth, *The fourth-order gravitational action for manifolds with boundaries,* Class. Quantum Grav. 2, 497-513, 1985.

[BGV] N. Berline, E. Getzler, M. Vergne, *Heat Kernals and Dirac Operators,* Springer-Verlag, Berlin, 1992.

[C1] A. Connes, *Quantized calculus and applications,* XIth International Congress of Mathematical Physics (paris,1994), 15-36, Internat Press, Cambridge, MA, 1995.

[C2] A. Connes. *The action functinal in noncommutative geometry,* Comm. Math. Phys., 117:673-683, 1998.

[FGLS] B. V. Fedosov, F. Golse, E. Leichtnam, and E. Schrohe. *The noncommutative residue for manifolds with boundary,* J. Funct. Anal, 142:1-31,1996.

[FGV] H. Figueroa, J. Gracia-Bondía, and J. Várilly, *Elements of Noncommutative Geometry,* Birkhäuser Boston 2001.

[GS] G. Grubb, E. Schrohe, *Trace expansions and the noncommutative residue for manifolds with boundary,* J. Reine Angew. Math., 536:167-207, 2001.

[Gu] V.W. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. 55 no.2, 131-160, 1985.

[H] S. W. Hawking, *General Relativity. An Einstein Centenary Survey,* Edited by S. W. Hawking and W. Israel, Cambridge University Press,Cambridge-New York, 1979.

[K] D. Kastler, *The Dirac operator and gravitiation,* Commun. Math. Phys, 166:633-643, 1995.

[KW] W. Kalau and M.Walze, *Gravity, non-commutative geometry, and the Wodzicki residue,* J. Geom. Phys., 16:327-344, 1995.

[RS] S. Rempel and B. W. Schulze, Index theory of elliptic boundary problems, Akademieverlag, Berlin, 1982.

[S] E. Schrohe, *Noncommutative residue, Dixmier's trace, and heat trace expansions on manifolds with boundary,* Contemp. Math. 242, 161-186, 1999.
[U] W. J. Ugalde, *Differential forms and the Wodzicki residue,* arXiv: Math, DG/0211361.
[Wa1] Y. Wang, *Differential forms and the Wodzicki residue for manifolds with boundary,* J. Geom. Phys., 56:731-753, 2006.
[Wa2] Y. Wang, *Differential forms and the noncommutative residue for manifolds with boundary in the non-product Case,* Lett. math. Phys., 77:41-51, 2006.
[Wo] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent.Math. 75 no.1 143-178, 1984.
[Y] Y. Yu, *The Index Theorem and The Heat Equation Method*, Nankai Tracts in Mathematics - Vol. 2, World Scientific Publishing, 2001.